12. Define dual space. Prove that  $\left(l_p^n\right)^* = l_q^n$  and  $\left(l_1^n\right)^* = l_\infty^n$  where :

$$l_p^n = \left\{ x = (x_1, x_2, \dots, x_n) : ||x|| = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \right\}$$

$$l_1^n = \left\{ x = (x_1, x_2, \dots, x_n) : ||x|| = \sum_{i=1}^n |x_i| \right\}$$

$$l_{\infty}^{n} = \left\{ x = (x_{1}, x_{2}, \dots, x_{n}) : ||x|| = \max_{1 \le i \le n} |x_{i}| \right\}$$

Or

If P is a projection on a Banach space B and if M and N are its range and null space, then prove that M and N are closed linear subspace of B such that  $B = M \oplus N$ . Also state and prove its converse part.

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Roll No. ..... Exam Code: J-19

## Subject Code—0366-X

## M. Sc. EXAMINATION

(Fourth Semester)

(Prior to 2011 Re-appear)

**MATHEMATICS** 

MAL-641

Functional Analysis

Time: 3 Hours Maximum Marks: 100

## **Section A**

**Note**: Attempt any *Seven* questions.  $7 \times 7 = 49$ 

- 1. Prove that a normed linear space X is complete iff every absolutely summable series in X is convergent (summable).
- 2. State and prove Minkowski's inequality for  $L^p$  space.

P.T.O.

- 3. Let M be a closed linear subspace of a normed linear space N and let  $x_0$  be a vector not in M, then there exists a functional F in N\* such that  $F(M) = \{0\}$  and  $F(x_0) \neq 0$
- **4.** Let X and Y be normed spaces over the field  $\mathbf{K}$  and  $T: X \xrightarrow{\text{onto}} Y$  be a linear operator. Then,  $T^{-1}$  exists and is a bounded linear operator iff  $\exists$  a constant K > 0 such that  $\|Tx\|_{\mathcal{Y}} \ge K \|x\|_{X}$ ,  $\forall x \in X$ .
- **5.** Prove that a linear transformation is closed iff its graph is a closed subspace.
- 6. Let X and Y be normed spaces and T : X → Y be linear operator. Then prove that T is compact iff it maps every bounded sequence <x<sub>n</sub>> in X onto a sequence <Tx<sub>n</sub>> in y which has a convergence.
- 7. State and prove Schwarz's inequality in an Inner product space.

- 8. Show that the linear space C[a, b] equipped with the norm given by  $||x||_{\infty} = \sup_{t \in [a, b]} |x(t)|, x \in C[a, b]$ , is not an inner product space and hence not a Hilbert space.
- 9. A subspace M of a Hilbert space H is closed in H iff  $M = M^{\perp \perp}$ .
- 10. If  $N_1$  and  $N_2$  are normal operators on a Hilbert space H with the property that either commutes with the adjoint of the other, then prove that  $N_1 + N_2$  and  $N_1N_2$  are normal.

## **Section B**

Note: Attempt all the questions.  $3\times17=17$ 

11. State and prove Riesz-Fisher theorem for the completeness of  $L^p$  space.

Or

State and prove Riesz-representation theorem for bounded linear functional on C[a, b].

- 13. Let H be a Hilbert space and let  $\langle e_i \rangle$  be an orthonormal set in H. Then the following conditions are all equivalent to each other:
  - (i)  $\langle e_i \rangle$  is complete
  - (ii)  $x \perp \langle e_i \rangle \Rightarrow x = 0$
  - (iii) If x is any arbitrary vector in H, then  $x = \sum (x, e_i)e_i$ .
  - (iv) If x is any arbitrary vector in H, then  $||x||^2 = \sum |(x, e_i)|^2$ .

Or

Let H be a Hilbert space and let f be an arbitrary functional in H\*. Then there exists a unique vector y in H such that f(x) = (x, y) for every x in H.

- 13. Let H be a Hilbert space and let  $\langle e_i \rangle$  be an orthonormal set in H. Then the following conditions are all equivalent to each other:
  - (i)  $\langle e_i \rangle$  is complete
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Or

Let H be a Hilbert space and let f be an arbitrary functional in H\*. Then there exists a unique vector y in H such that f(x) = (x, y) for every x in H.